3d Quantum Field Theories and Langlands duality

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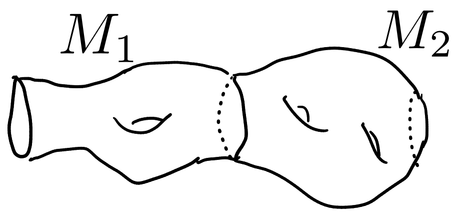
Reference : arXiv:2409.06303

BenZvi-Sakellaridis-Venkatesh arXiv:2409.04677

Axiomatic approach to (topological) Quantum Field Theories (Atiyah)

A d-dim Topological Quantum Field Theory Z is an `assignement':

1. $(d-1)$ dim closed manifold $X \rightsquigarrow$ a (complex) vector space $Z(X)$
   (quantum Hilbert space)
2. d-dim manifold M with boundary $\partial M \rightsquigarrow$ a vector $Z(M) \in Z(\partial M)$

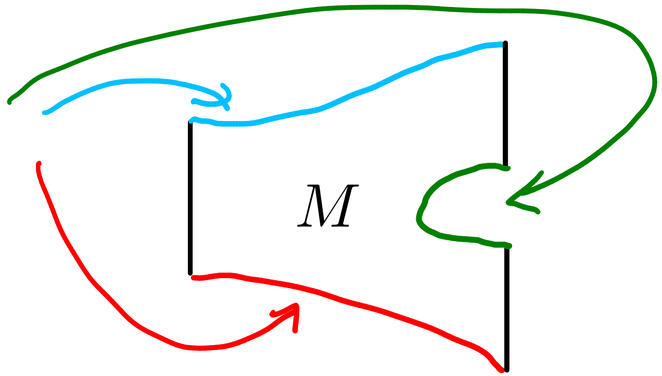
such that
- $Z(X_1 \sqcup X_2) = Z(X_1) \otimes_{\mathbb{C}} Z(X_2)$, $Z(-X) = Z(X)^*$
- 

  $\Rightarrow \quad Z(M_1 \cup_X M_2) = Z(M_1) \circ Z(M_2)$

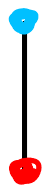
<u>Remark</u>. If Z is defined for arbitrary d-dim manifolds, $Z(X)$ is automatically finite dimensional . This is too restrictive for our purpose. We will use TQFT in an informal way.

Extended Topological Quantum Field Theories : Allow codim 1 `Defects'

Impose bdry conditions
(Dirichlet, Neumann, ...)



Example : Consider QFT of the space of all maps $f : M \to T$
to a target space T . We choose a submanifold $S \subset T$
and consider f such that $f(\text{horizontal bdry}) \subset S$.

All bdry conditions form a category:  $= \mathrm{Hom}(\bullet, \bullet)$

Above picture = composite of hom's.

cf. Fukaya category of a symplectic manifold

Kapustin-Witten '07

Geometric Langlands is a consequence of S-duality of 4d SUSY $\mathbf{G}$- Yang-Mills theory for $M^4 = \bullet \times C^2$ where C is a Riemann surface and $\mathbf{G}$ is a complex reductive group.

Categories of bdry conditions, after `twisting', are relevant ones for geometric Langlands: $\mathrm{Fuk}(T^*\mathrm{Bun}_{\mathbf{G}}(C)) \cong \mathrm{Shv}(\mathrm{Bun}_{\mathbf{G}}(C))$

$D(\mathrm{Coh}(\mathrm{Loc}_{\mathbf{G}}(C)))$

Then S-duality claims

$$\mathrm{Fuk}(T^*\mathrm{Bun}_{\mathbf{G}}(C)) \cong D(\mathrm{Coh}(\mathrm{Loc}_{\mathbf{G}^\vee}(C)))$$

where $\mathbf{G}^\vee$ is the Langlands dual group.

Point: when C is small, 4d YM theory is 2d σ - model whose target space is Hitchin moduli space $\mathrm{Hit}_{\mathbf{G}}(C)$ on C .

S-duality becomes 2d (homological) mirror symmetry.

Gaiotto-Witten '09 (cf. Gaiotto '18)

- 3d SUSY QFT's give a class of bdry conditions.

- S-duality is related to 3d mirror symmetry of 3d SUSY QFT's.

An example of 3d SUSY QFT

$\mathbf{G} \curvearrowright \mathbf{M}$: hamiltonian action of a complex reductive group on a smooth affine algebraic symplectic manifold (with a condition, called anomally free)

Therefore we expect

$\mathbf{G} \curvearrowright \mathbf{M}$ $\rightsquigarrow$ An object $\mathcal{L}(C, \mathbf{G} \curvearrowright \mathbf{M})$ in $\mathrm{Fuk}(T^*\mathrm{Bun}_{\mathbf{G}}(C))$

$\rightsquigarrow$ An object $\mathcal{V}(C, \mathbf{G} \curvearrowright \mathbf{M})$ in $D(\mathrm{Coh}(\mathrm{Loc}_{\mathbf{G}}(C)))$

Moreover, the S-dual bdry condition also comes from another hamiltonian space $\mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee$ in many examples. Or one can at least \`approximate' the dual by $\mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee$. (Coulomb branches)

A Conjectural Mathematical Consequence :

$$\mathrm{Fuk}(T^*\mathrm{Bun}_{\mathbf{G}}(C)) \ni \mathcal{L}(C, \mathbf{G} \curvearrowright \mathbf{M}) \longleftrightarrow \mathcal{V}(C, \mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee) \in D(\mathrm{Coh}(\mathrm{Loc}_{\mathbf{G}^\vee}(C)))$$

$$D(\mathrm{Coh}(\mathrm{Loc}_{\mathbf{G}}(C))) \ni \mathcal{V}(C, \mathbf{G} \curvearrowright \mathbf{M}) \longleftrightarrow \mathcal{L}(C, \mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee) \in \mathrm{Fuk}(T^*\mathrm{Bun}_{\mathbf{G}^\vee}(C))$$

under geometric Langlands correspondence.

Example (conjecture) $\mathbf{G} \curvearrowright \mathbf{M} = \mathbf{G} \times \mathcal{S} \longleftrightarrow \mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee = \{0\}$

(equivariant Kostant-Slodowy slice)

$\rightsquigarrow$ $\quad \text{Hitchin section} \longleftrightarrow \mathcal{O}_{\mathrm{Loc}_{\mathbf{G}^\vee}(C)}$

$? \longleftrightarrow 0 \text{ section} \subset T^*\mathrm{Bun}_{\mathbf{G}^\vee}(C)$

under geometric Langlands

Examples (1) $\mathbb{C}^\times \curvearrowright T^*\mathbb{C}^\times \longleftrightarrow \mathbb{C}^\times \curvearrowright \{0\}$

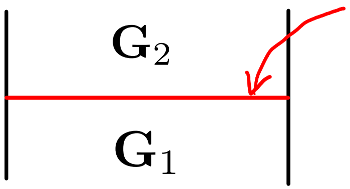
(2) $\mathbb{C}^\times \curvearrowright \mathbb{C}^2 \longleftrightarrow \mathbb{C}^\times \curvearrowright \mathbb{C}^2$
weight 1

More generally $\mathbb{C}^\times \curvearrowright \mathbb{C}^{2\ell} \longleftrightarrow \mathbb{C}^\times \curvearrowright \mathbb{C}^2/(\mathbb{Z}/\ell)$
weight 1

(3) $\mathbb{C}^\times \curvearrowright \mathbb{C}^2 \longleftrightarrow \mathbb{C}^\times \curvearrowright \mathbb{C}^2/(\mathbb{Z}/\ell)$
weight ℓ

Important generalization of bdry conditions : Interface

$\mathbf{G}_1, \mathbf{G}_2$: pair of cpx reductive groups

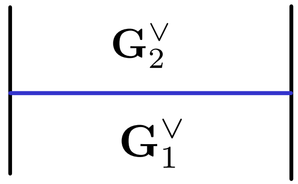


3d QFT with

$\mathbf{G}_1 \times \mathbf{G}_2$ symmetry

e.g. $\mathbf{G}_1 \curvearrowright \mathbf{M} \curvearrowleft \mathbf{G}_2$

$\longleftrightarrow$ S-dual



$\mathbf{G}_1^\vee \curvearrowright \mathbf{M}^\vee \curvearrowleft \mathbf{G}_2^\vee$

This gives pairs of functors, intertwined by Langlands correspondence

$$\begin{array}{ccc} \mathrm{Fuk}(T^*\mathrm{Bun}_{\mathbf{G}_2}(C)) & \cong & D(\mathrm{Coh}(\mathrm{Loc}_{\mathbf{G}_2}(C))) \\ \uparrow & & \uparrow \\ \mathrm{Fuk}(T^*\mathrm{Bun}_{\mathbf{G}_1}(C)) & \cong & D(\mathrm{Coh}(\mathrm{Loc}_{\mathbf{G}_1}(C))) \end{array}$$

Trivial Example : $\mathbf{G} \curvearrowright T^*\mathbf{G} \curvearrowleft \mathbf{G} \quad \longleftrightarrow \quad \mathbf{G}^\vee \curvearrowright T^*\mathbf{G}^\vee \curvearrowleft \mathbf{G}^\vee$

Non - Example :(Eisenstein series) $\mathbf{T} \curvearrowright T^*(\mathbf{G}/\mathbf{U}) \curvearrowleft \mathbf{G} \longleftrightarrow \mathbf{T}^\vee \curvearrowright T^*(\mathbf{G}^\vee/\mathbf{U}^\vee) \curvearrowleft \mathbf{G}^\vee$

Example: $\mathrm{SO}_{2n} \curvearrowright \mathbb{C}^{2n} \otimes \mathbb{C}^{2m} \curvearrowleft \mathrm{Sp}_{2m} \longleftrightarrow \mathrm{SO}_{2n} \curvearrowright \text{equivariant slice} \curvearrowleft \mathrm{SO}_{2m+1}$

(orthosymplectic quiver)

Relative Langlands Duality Conjecture (Ben-Zvi, Sakellaridis, Venkatesh)

These relations hold at all `tiers' of the Langlands duality (global, local, geometric, arithmetic, etc)

Arithmetic: $\mathcal{L}(C, \mathbf{G} \curvearrowright \mathbf{M})$, $\mathcal{V}(C, \mathbf{G} \curvearrowright \mathbf{M})$ are `function' on moduli spaces.

Local : $C = D^*$ (formal punctured disk) ⇒ one upper categorical level

Geometric Langlands predicts an equivalence of 2-categories.
Hence $\mathcal{L}(D^*, \mathbf{G} \curvearrowright \mathbf{M})$, $\mathcal{V}(D^*, \mathbf{G} \curvearrowright \mathbf{M})$ are categories.

Then $\mathcal{L}(D^*, \mathbf{G} \curvearrowright \mathbf{M}) \longleftrightarrow \mathcal{V}(D^*, \mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee)$ means a categorical equiv.

Note that $\mathbf{G} \curvearrowright \mathbf{M} \longleftrightarrow \mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee$ is independent of C , or all tiers.

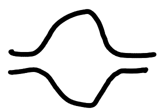
Question. How do we compute $\mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee$ from $\mathbf{G} \curvearrowright \mathbf{M}$?

Gaiotto-Witten : This is a `cousin' of 3D mirror symmetry
BZSV : Use local equivalence to recover $\mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee$

Either way, $\mathbf{M}^\vee$ is analog of Coulomb branch of a 3d QFT theory
associated with $\mathbf{G} \curvearrowright \mathbf{M}$,
defined mathematically by Braverman-Finkelberg-N.

I do not review the detail. Very roughly (after assuming $\mathbf{M} = T^*\mathbf{N}$)

From $\mathbf{G} \curvearrowright \mathbf{M}$, we construct a moduli space of $\mathbf{G}$ - bundles and $\mathbf{N}$-valued sections over the raviolo space $D \cup_{D^*} D$.



Next we define a commutative ring from homology of this moduli sp.

Finally we recover $\mathbf{M}^\vee$ as $\mathbb{C}[\mathbf{M}^\vee] =$ the commutative ring.

Questions. (1) How to define $\mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee$ for more general $\mathbf{G} \curvearrowright \mathbf{M}$,
not necessarily $\mathbf{M} = T^*\mathbf{N}$?

(2) When $\mathbf{M}$ is singular ?

e.g. $\mathbf{G} \curvearrowright T^*\mathbf{G} \rightsquigarrow \mathbf{G}^\vee \curvearrowright \mathcal{N}_{\mathbf{G}^\vee}$ (nilpotent cone)

How to define the S-dual of $\mathcal{N}_{\mathbf{G}^\vee}$?

(3) Is this a duality ? Namely is $\mathbf{G} \curvearrowright \mathbf{M}$ the S-dual of $\mathbf{G}^\vee \curvearrowright \mathbf{M}^\vee$?
At this moment, we only check this by calculating the S-dual.

Thank you very much !